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J. VAN MILL & A. SCHRIJVER

SUPEREXTENSIONS WHICH ARE HILBERT CUBES

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Superextensions which are Hilbert cubes*)

by

J. van Mill & A. Schrijver

ABSTRACT

It is shown that each separable metric, not totally disconnected, topological space admits a superextension homeomorphic to the Hilbert cube. Moreover, for simple spaces, such as the closed unit interval or the n-spheres S_n, we give easily described subbases for which the corresponding superextension is homeomorphic to the Hilbert cube.

KEY WORDS & PHRASES: *superextension, Hilbert cube, Z-set, convex.*

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1. INTRODUCTION

In [5], DE GROOT defined a space X to be *supercompact* provided that it possesses a binary closed subbase, i.e. a closed subbase $\mathcal{S}$ with the property that if $\mathcal{S}' \subset \mathcal{S}$ and $\cap\mathcal{S}' = \emptyset$ then there exist $S_0, S_1 \in \mathcal{S}'$ such that $S_0 \cap S_1 = \emptyset$. Clearly, according to the lemma of ALEXANDER, every supercompact space is compact. The class of supercompact spaces contains the compact orderable spaces, compact tree-like spaces (BROUWER & SCHRIJVER [3], VAN MILL [8]) and compact metric spaces (STROK & SZYMAŃSKI [12]). Moreover, there are compact Hausdorff spaces which are not supercompact (BELL [2], VAN MILL [10]). There is a connection between supercompact spaces and graphs (see e.g., DE GROOT [6], BRUIJNING [4], SCHRIJVER [11]); moreover, supercompact spaces can be characterized by means of so-called interval structures (BROUWER & SCHRIJVER [3]).

Let X be a T_1-space and $\mathcal{S}$ a closed T_1-subbase for X (a closed subbase $\mathcal{S}$ for X is called T_1 if for all $S \in \mathcal{S}$ and $x \in X$ with $x \notin S$, there exists an $S_0 \in \mathcal{S}$ with $x \in S_0$ and $S_0 \cap S = \emptyset$). The *superextension* $\lambda_{\mathcal{S}}(X)$ *of* X *relative the subbase* $\mathcal{S}$ is the set of all maximal linked systems $\mathcal{M} \subset \mathcal{S}$ (a subsystem of $\mathcal{S}$ is called *linked* if every two of its members meet; a *maximal linked system* or *mls* is a linked system not properly contained in another linked system) topologized by taking $\{\{\mathcal{M} \in \lambda_{\mathcal{S}}(X) \mid S \in \mathcal{M}\} \mid S \in \mathcal{S}\}$ as a closed subbase. Clearly, this subbase is binary, hence $\lambda_{\mathcal{S}}(X)$ is supercompact, while moreover X can be embedded in $\lambda_{\mathcal{S}}(X)$ by the natural embedding $i\colon X \to \lambda_{\mathcal{S}}(X)$ defined by $i(x) := \{S \in \mathcal{S} \mid x \in S\}$. VERBEEK's monograph [12] is a good place to find the basic theorems about superextensions. In this paper we will show that for many spaces there are superextensions homeomorphic to the Hilbert cube Q; moreover for simple spaces such as the unit interval or the n-spheres S_n we will present easily described subbases for which the corresponding superextension is homeomorphic to Q. Here, a classical theorem of KELLER [7], which says that *each infinite-dimensional compact convex subset of the separable Hilbert space is homeomorphic to* Q, is of great help.

2. SOME EXAMPLES

In this section we will give some examples. If X is an ordered space, then the Dedekind completion of X will be denoted by $\overline{X}$. Roughly speaking, $\overline{X}$ can be obtained from X by filling up every gap. We define $\overline{\overline{X}}$ to be that ordered space which can be obtained from X by filling up every gap with two points, except for possible endgaps, which we supply with one point. The compact space $\overline{\overline{X}}$ thus obtained, clearly contains X as a dense subspace. Define

$$\mathcal{G}_1 = \{A \subset X \mid \exists\, x \in X: A = (\leftarrow, x] \text{ or } A = [x, \rightarrow)\}$$

and

$$\mathcal{T}_1 = \{A \subset X \mid A \text{ is a closed half-interval}\}$$

(as usual, a half-interval is a subset $A \subset X$ such that either for all $a,b \in X$: if $b \leq a \in A$ then $b \in A$, or for all $a,b \in X$: if $b \geq a \in A$ then $b \in A$)

and

$$\mathcal{T}_2 = \{A \subset X \mid \exists\, A_0, A_1 \in \mathcal{T}_1: A = A_0 \cup A_1 \text{ or } A = A_0 \cap A_1\},$$

respectively.

Notice that $\mathcal{G}_1$ equals $\mathcal{T}_1$ in case X is compact or connected. It is easy to see that $\lambda_{\mathcal{G}_1}(X) \cong \overline{X}$ and that $\lambda_{\mathcal{T}_1}(X) \cong \overline{\overline{X}}$.

What about $\lambda_{\mathcal{T}_2}(X)$?

Example (i) If $X = I$, then $\lambda_{\mathcal{G}_1}(X) = \lambda_{\mathcal{T}_1}(X) \cong I$. On the other hand $\lambda_{\mathcal{T}_2}(X)$ is homeomorphic to the Hilbert cube Q (see section 4).

(ii) If $X = \mathbb{Q}$, then $\lambda_{\mathcal{G}_1}(X) \cong I$ and $\lambda_{\mathcal{T}_1}(X)$ is a non-metrizable separable compact ordered space, which has much in common with the well-known Alexandroff double of the closed unit interval. In this case, $\lambda_{\mathcal{T}_2}(X)$ is a compact totally disconnected perfect space of weight $2^{\aleph_0}$. (The total disconnectedness of $\lambda_{\mathcal{T}_2}(X)$ follows from the following observation: for every $T_0, T_1 \in \mathcal{T}_2$ with $T_0 \cap T_1 = \emptyset$ there exists a $T_0' \in \mathcal{T}_2$ such that $T_0 \subset T_0'$ and $T_0 \cap T_1 = \emptyset$ and $X \setminus T_0' \in \mathcal{T}_2$. For every finite linked system $\{X \setminus T_i \mid T_i \in \mathcal{T}_2,\ i \in \{1,2,\dots,n\}\}$ it is easy to construct two distinct mls's L_0 and L_1 belonging to $\bigcap_{i=1}^{n} \{M \in \lambda_{T_2}(X) \mid T_i \notin M\}$ showing that $\lambda_{T_2}(X)$ is perfect. Finally λ_{T_1} can be embedded in $\lambda_{T_2}(X)$; hence weight $(\lambda_{T_2}(X)) = 2^{\aleph_0}$.)

(iii). If $X = \mathbb{R}\setminus\mathbb{Q}$, then $\lambda_{G_1}(X) \cong I$, while $\lambda_{T_1}(X) \cong \lambda_{T_2}(X) \cong C$, the Cantor discontinuum, for it is easy to see that $\lambda_{T_1}(X)$ and $\lambda_{T_2}(X)$ both are totally disconnected compact metric perfect spaces.

Finally define

$$G_2 = \{A \subset X \mid \exists\, A_0, A_1 \in G_1 : A = A_0 \cup A_1 \quad \text{or} \quad A = A_0 \cap A_1\}.$$

Notice that G_2 equals T_2 in case X is compact or connected.

Example (i) If $X = I$, then $\lambda_{G_2}(X) \cong Q$ (section 4).

(ii) If $X = \mathbb{Q}$, then $\lambda_{G_2}(X) \cong Q$.

(iii) If $X = \mathbb{R}\setminus\mathbb{Q}$, then $\lambda_{G_2}(X) \cong Q$.

The fact that $\lambda_{G_2}(\mathbb{Q}) \cong \lambda_{G_2}(\mathbb{R}\setminus\mathbb{Q}) \cong Q$ can be derived from the result $\lambda_{G_2}(I) \cong Q$. To see this, define

$$G_2' = \{A \subset I \mid A \in G_2 \text{ and A has rational endpoints}\}$$

and

$$G_2'' = \{A \subset I \mid A \in G_2 \text{ and A has irrational endpoints}\}.$$

By theorem 5 and theorem 7 of [9] (cf. theorem 3.1 below), it follows that

$$\lambda_{G_2}(I) \cong \lambda_{G_2'}(I) \cong \lambda_{G_2}(\mathbb{Q})$$

and

$$\lambda_{G_2}(I) \cong \lambda_{G_2''}(I) \cong \lambda_{G_2}(\mathbb{R}\setminus\mathbb{Q}).$$

3. SUPEREXTENSIONS WHICH ARE HILBERT CUBES

In this section we will show that for each separable metric, not totally disconnected topological space X, there exists a normal closed T_1-subbase S such that $\lambda_S(X)$ is homeomorphic to the Hilbert cube Q. First we will give some preliminary definitions and recapitulate some well-known results from the literature, which are needed in the remainder of this section. A closed subset B of Q is called a *Z-set* ([1]), if for each $\varepsilon > 0$ there exists a map $f: Q \to Q\setminus B$ such that $d(f,id) < \varepsilon$. Examples of Z-sets are compact subsets of $(0,1)^\infty$ and closed subsets of Q which project onto a point in infinitely many coordinates. In fact, Z-sets can be characterized by the property that for every Z-set B there exists an autohomeomorphism ϕ of Q which maps B onto a set which projects onto a point in infinitely many coordinates ([1]). Obviously, the property of being a Z-set is a topological invariant. Moreover, it is easy to show that a closed countable union of Z-sets is again a Z-set. The importance of Z-sets is illustrated by the following theorem due to ANDERSON [1].

THEOREM. *Any homeomorphism between two* Z-*sets in* Q *can be extended to an autohomeomorphism of* Q.

We will apply this theorem to show that every separable metric, not totally disconnected topological space X can be embedded in Q in such a way that Q has the structure of a superextension of X, i.e. every point of Q represents an mls in a suitable closed subbase for X. The canonical binary subbase for Q is

$$T = \{A \subset Q \mid A = \pi_n^{-1}[0,x] \quad \text{or} \quad A = \pi_n^{-1}[x,1], \text{ with } n \in \mathbb{N} \text{ and } x \in I\}$$

and consequently, if we embed X in Q in such a way that for every two elements $T_0,T_1 \in T$ with $T_0 \cap T_1 \neq \emptyset$ we have that $T_0 \cap T_1 \cap X \neq \emptyset$, then Q is a superextension of X; this is a consequence of the following theorem ([9], theorem 5).

THEOREM 3.1. *Let* X *be a subspace of the topological space* Y. *Then* Y *is homeomorphic to a superextension of* X *if and only if* Y *possesses a binary closed subbase* T *such that for all* $T_0,T_1 \in T$ with $T_0 \cap T_1 \neq \emptyset$ *we have that* $T_0 \cap T_1 \cap X \neq \emptyset$.

In particular, in theorem 3.1 $Y \cong \lambda_{\mathcal{T}\cap X}(X)$, where $\mathcal{T} \cap X = \{T \cap X \mid T \in \mathcal{T}\}$.

THEOREM 3.2. *For every separable metric, not totally disconnected topological space* X, *there exists a normal closed* T_1*-subbase* S *such that* $\lambda_S(X)$ *is homeomorphic to the Hilbert cube* Q.

PROOF. Assume that X is embedded in $Q(=I^{\mathbb{N}})$ and let C be a non-trivial compo- of X. Choose a convergent sequence B in C. Furthermore, define a sequence $\{y_n\}_{n=0}^{\infty}$ in Q by

$$(y_n)_i = \begin{cases} 1 & \text{if} \quad i \neq n \\ 0 & \text{if} \quad i = n, \end{cases}$$

for $i = 1,2,\ldots,$.

It is clear that

$$\lim_{n\to\infty} y_n = y_0.$$

Moreover define $z \in Q$ by $z_i = 0$ $(i=1,2,\ldots,)$. Then

$$E = \{y_n \mid n \in \mathbb{N}\} \cup \{z\}$$

is a convergent sequence and therefore is homeomorphic to B. Since B and E both are closed countable unions of Z-sets in Q, they themselves are Z-sets. Choose a homeomorphism $\phi\colon B \to E$ and extend this homeomorphism to an autohomeomorphism of Q. This procedure shows that we may assume that X is embedded in Q in such a way that $E \subset C$. Let $T_0, T_1 \in \mathcal{T}$ such that $T_0 \cap T_1 \neq \emptyset$, where $\mathcal{T}$ is the canonical binary closed subbase for Q. We need only consider the following 4 cases:

CASE 1: $T_0 = \pi_{n_0}^{-1}[0,x]$; $T_1 = \pi_{n_0}^{-1}[y,1]$ $(x \geq y)$.
Since $z \in T_0$ and $y_0 \in T_1$ and C is connected, it follows that $\phi \neq T_0 \cap T_1 \cap C \subset T_0 \cap T_1 \cap X$.

CASE 2: $T_0 = \pi_{n_0}^{-1}[0,x]$; $T_1 = \pi_{n_1}^{-1}[y,1]$ $(n_0 \neq n_1)$.
Then $y_{n_0} \in T_0 \cap T_1 \cap X$.

CASE 3: $T_0 = \Pi_{n_0}^{-1}[0,x]$; $T_1 = \Pi_{n_1}^{-1}[0,y]$.

Then $z \in T_0 \cap T_1 \cap X$.

CASE 4: $T_0 = \Pi_{n_0}^{-1}[x,1]$; $T_1 = \Pi_{n_1}^{-1}[y,1]$.

Then $y_0 \in T_0 \cap T_1 \cap X$.

This completes the proof of the theorem. □

4. A SUPEREXTENSION OF THE CLOSED UNIT INTERVAL

In the present section we will prove that $\lambda_{G_2}(I)$ is homeomorphic to the cube, where $G_2 = \{[x,y] \mid x,y \in I\} \cup \{[0,x] \cup [y,1] \mid x,y \in I\}$. For this purpose we introduce

$$F = \{f:I \to I \mid f(0) = 0 \text{ and if } x,y \in I \text{ and } x \le y \text{ then } 0 \le f(y)-f(x) \le y - x\}.$$

Hence each $f \in F$ is continuous and monotone non-decreasing. On F we define a topology by considering F as a subspace of $C[I,I]$ with the point-open topology. We obtain the same topology on F by ordering F partially as follows:

$$f \le g \text{ iff for each } x \in I: f(x) \le g(x), \quad (f,g \in F),$$

and then taking as a closed subbase for F the collection of all subsets of the form $\{f \in F \mid f \le f_0\}$ or $\{f \in F \mid f \ge f_0\}$, where f_0 runs through F. We first prove that $F \cong Q$ and next that $\lambda_{G_2}(I) \cong F$; we conclude that $\lambda_{G_2}(I) \cong Q$.

THEOREM 4.1. $F \cong Q$.

PROOF. We show that F is a compact, infinite-dimensional, convex subspace of I^I, with countable base; hence, by KELLER's theorem, F is homeomorphic to the Hilbert cube Q.

F is clearly a convex subspace of I^I; it is also clear that $(F,\le)$, as defined above, is a complete lattice, whence F is compact. F has a countable subbase, since the collection of all subsets of the forms $\{f \in F \mid f(x) \ge y\}$ and $\{f \in F \mid f(x) \le y\}$ where $x,y \in \mathbb{Q} \cap I$, forms a countable closed subbase for F.

Finally, F is infinite-dimensional, because Q can be embedded in F.

For, let $\underline{a} = (a_1, a_2, a_3, \ldots) \in I^{\mathbb{N}}$. Let $G(\underline{a})$ be the smallest function f in F (in the ordering $\leq$ of F) such that for each i = 1,2,3,... the following holds:

$$f(\frac{3}{2^{i+1}}) \geq \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} a_i.$$

It can be seen easily that G defines a topological embedding of Q in F. □

THEOREM 4.2. $\lambda_{G_2}(I) \cong F$.

PROOF. Define a function $K: \lambda_{G_2}(I) \to I$ by:

$$K(M) = \inf\{x \in I \mid [0,x] \in M\}, \qquad (M \in \lambda_{G_2}(I)),$$

and a function $H: \lambda_{G_2}(I) \to F$ by:

$$H(M)(i) = \inf\{x \in I \mid [0,x] \cup [y,1] \in M,\ x + y = K(M) + i\},\ (i \in I,\ M \in \lambda_{G_2}(I)).$$

We prove that H is an homeomorphism between $\lambda_{G_2}(I)$ and F.
First we observe that:

$$\begin{aligned}
&K(M) \leq x \quad \text{iff} \quad [0,x] \in M;\\
&K(M) \geq x \quad \text{iff} \quad [x,1] \in M;\\
&K(M) = x \quad \text{iff} \quad [0,x] \in M \text{ and } [x,1] \in M;\\
&H(M)(i) \leq x \quad \text{iff} \quad [0,x] \cup [K(M) + i - x,1] \in M;\\
&H(M)(i) \geq x \text{ iff } [x,K(M) + i - x] \in M;\\
&H(M)(i) = x \quad \text{iff} \quad [0,x] \cup [K(M) + i - x,1] \in M \text{ and}\\
&\qquad\qquad\qquad\qquad [x,K(M) + i - x] \in M;
\end{aligned}$$

these facts follow easily from the fact that M is a maximal linked system in G_2. Also we have $K(M) = H(M)(1)$.

Next we show that $H(M) \in F$, for each maximal linked system M. In fact (i) $H(M)(0) = 0$, for $[0,0] \cup [K(M),1] \in M$ and $[0,K(M)] \in M$; (ii) if $i \leq j$, $H(M)(i) = x$, $H(M)(j) = y$, then $x \leq y$, for $[x,K(M) + j - x] \supset [x,K(M) + i - x] \in M$, hence $[x,K(M) + j - x] \in M$ and $y = H(M)(j) \geq x$; also $y - x \leq j - i$, for $[y - j + i, K(M) + i - (y-j+i)] \supset [y, K(M) + j - y] \in M$, hence $x = H(M)(i) \geq y - j + i$.

H is a one-to-one function, for suppose $M_1, M_2 \in \lambda_{G_2}(I)$, $M_1 \neq M_2$ and $H(M_1) = H(M_2)$. Let $a = K(M_1) = H(M_1)(1) = H(M_2)(1) = K(M_2)$, i.e. $[0,a] \in M_1 \cap M_2$ and $[a,1] \in M_1 \cap M_2$. Since $M_1 \neq M_2$ we may suppose that there are x' and y' such that $[0,x'] \cup [y',1] \in M_1 \setminus M_2$. Since $[0,a] \in M_2$ and $[a,1] \in M_2$, we have $x' < a < y'$. Let $i = x' + y' - a \in [x',y'] \subset I$. Then since $[0,x'] \cup [a+i-x',1] = [0,x'] \cup [y',1] \in M_1 \setminus M_2$, we find that $H(M_1)(i) \leq x' < H(M_2)(i)$ and this is a contradiction. H is also a surjection. Take $f \in F$ and let:

$$L = \{[f(i), f(1) + i - f(i)] \mid i \in I\} \cup \{[0,f(i)] \cup [f(1) + i - f(i),1] \mid i \in I\}.$$

Then by definition of F, it is easy to see that L is a linked system in G_2. L is contained in some maximal linked system M of G_2, and for this M it holds that $K(M) = f(1)$ while for each $i \in I$: $H(M)(i) = f(i)$; i.e. $H(M) = f$. Finally we prove that H is continuous. Let $i,x \in I$. Then

$$\{M \in \lambda_{G_2}(I) \mid H(M)(i) \leq x\} = \bigcap_{y \in I}\{M \in \lambda_{G_2}(I) \mid [0,x] \cup [y,1] \in M \text{ or } [0, x + y - i] \in M\},$$

and hence this set is closed. For, let $M \in \lambda_{G_2}(I)$ such that $H(M)(i) \leq x$; this last inequality means that $[0,x] \cup [K(M) + i - x, 1] \in M$. If $y \geq K(M) + i - x$, then $[0, y + x - i] \supset [0, K(M)] \in M$; if $y \leq K(M) + i - x$ then $[0,x] \cup [y,1] \supset [0,x] \cup [K(M) + i - x, 1] \in M$.
Conversely, suppose that

$$[0,x] \cup [y,1] \in M \text{ or } [0, x + y - i] \in M,$$

for each $y \in I$, then also $[0, x + y - i] \notin M$ for each $y < K(M) + i - x$; hence $[0,x] \cup [y,1] \in M$; we conclude that $[0,x] \cup [K(M) + i - x, 1] \in M$, i.e. $H(M)(i) \leq x$.

In the same way one proves:

$$\{M \in \lambda_{G_2}(I) \mid H(M)(i) \geq x\} = \bigcap_{y \in I}\{M \in \lambda_{G_2}(I) \mid [x,y] \in M \text{ or } [x+y-i,\ 1] \in M\},$$

and hence is closed. □

As a consequence of these two theorems we have, as announced,

THEOREM: 4.3. $\lambda_{G_2}(I) \cong Q$.

5. A SUPEREXTENSION OF THE n-SPHERE

In this final section we show that the superextension of the n-sphere S^n with respect to the collection of all closed massive n-balls in S^n is homeomorphic with the Hilbert-cube. As usual, the n-spheres S^n is the space

$$\left\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i^2 = 1\right\}$$

and the closed massive n-ball with centre $\underline{x} \in S^n$ and radius $\varepsilon \geq 0$ is the set

$$B(\underline{x},\varepsilon) = \{\underline{y} \in S^n \mid d(\underline{x},\underline{y}) \leq \varepsilon\}.$$

Writing $\mathcal{B}$ for the collection of all closed massive n-balls in S^n, we will prove that, if $n \geq 1$, $\lambda_{\mathcal{B}}(S^n) \cong Q$. Obviously $\lambda_{\mathcal{B}}(S^1)$ is the superextension of the circle with respect to the set of closed intervals. For the definition of $\mathcal{B}$ it does not matter whether the euclidian metric of $\mathbb{R}^{n+1}$ or the sphere metric of S^n (in this case the distance between $\underline{x}$ and $\underline{y}$ in S^n is arccos $\Sigma_{i=0}^{n} x_i y_i$, i.e. the minimum length of a curve between $\underline{x}$ and $\underline{y}$ on S^n) is used. However, in the proof of the theorem we need the latter metric and we call this metric d. Furthermore we define, for each point $\underline{x} = (x_0, x_1, \ldots, x_n) \in S^n$, the *antipode* $\bar{\underline{x}}$ of $\underline{x}$ by $\bar{\underline{x}} = (-x_0, -x_1, \ldots, -x_n)$.

THEOREM 5.1. *If* $n \geq 1$, $\lambda_{\mathcal{B}}(S^n)$ *is homeomorphic to the Hilbert-cube* Q.

PROOF. In fact we show that $\lambda_{\mathcal{B}}(S^n)$ is compact and infinite-dimensional and has a countable base and that $\lambda_{\mathcal{B}}(S^n)$ can be embedded as a convex subspace in $\mathbb{R}^{S^n}$; hence, by KELLER's theorem, $\lambda_{\mathcal{B}}(S^n)$ is homeomorphic to Q.
Clearly, $\lambda_{\mathcal{B}}(S^n)$ is compact.

To prove that $\lambda_{\mathcal{B}}(S^n)$ has a countable base, let X be a countable dense subset of S^n. Define $\mathcal{B}_0 = \{B(\underline{x},\varepsilon) \mid \underline{x} \in X,\ \varepsilon \in \mathbb{Q},\ \varepsilon \geq 0\}$. It is not difficult to see that $P\colon \lambda_{\mathcal{B}}(S^n) \to \lambda_{\mathcal{B}_0}(S^n)$, such that $P(M) = M \cap \mathcal{B}_0$ $(M \in \lambda_{\mathcal{B}}(S^n))$ is a homeomorphism; hence, since $\lambda_{\mathcal{B}_0}(S^n)$ has a countable base, $\lambda_{\mathcal{B}}(S^n)$ also has a countable base. Next, $\lambda_{\mathcal{B}}(S^n)$ is infinite-dimensional, since $\lambda_{G_2}(I)(\simeq Q)$ can be embedded in $\lambda_{\mathcal{B}}(S^n)$. For, let

$$Y = \{\underline{x} \in S^n \mid \underline{x} = (x_0,x_1,\ldots,x_n),\ x_1 \geq 0,\ x_2 = \ldots = x_n = 0\};$$

this subspace is homeomorphic to I. Let G_2 be as defined in section 3, i.e. G_2 is the collection of all closed subsets Y' of Y such that Y' is connected or $Y \setminus Y'$ is connected. Define $T\colon \lambda_{G_2}(Y) \longrightarrow \lambda_{\mathcal{B}}(S^n)$ by $T(M) = \{B \in \mathcal{B} \mid B \cap Y \in M\}$, $(M \in \lambda_{G_2}(I))$. Again it is not difficult to prove that T is a topological embedding. Hence $\lambda_{G_2}(I) \simeq Q$ can be embedded in $\lambda_{\mathcal{B}}(S^n)$, i.e. $\lambda_{\mathcal{B}}(S^n)$ is infinite-dimensional.

Finally we embed $\lambda_{\mathcal{B}}(S^n)$ as a convex subspace in $\mathbb{R}^{S^n}$, by means of the function $U\colon \lambda_{\mathcal{B}}(S^n) \longrightarrow \mathbb{R}^{S^n}$, determined by:

$$U(M)(\underline{x}) = \inf\{\varepsilon \geq 0 \mid B(\underline{x},\varepsilon) \in M\},\ (M \in \lambda_{G_2}(S^n),\ \underline{x} \in S^n).$$

The mapping U is continuous and one-to-one since $U(M)(\underline{x}) \leq \varepsilon$ iff $B(\underline{x},\varepsilon) \in M$, and $U(M)(\underline{x}) \geq \varepsilon$ iff $B(\bar{\underline{x}},\pi-\varepsilon) \in M$. And indeed, $U[\lambda_{\mathcal{B}}(S^n)]$ is a convex subspace of $\mathbb{R}^{S^n}$. In order to show this, we need only prove: if $M_1, M_2 \in \lambda_{\mathcal{B}}(S^n)$, then there exists an $M \in \lambda_{\mathcal{B}}(S^n)$ such that $U(M) = \frac{1}{2}U(M_1) + \frac{1}{2}U(M_2)$ $(U[\lambda_{\mathcal{B}}(S^n)]$ being compact and hence closed in $\mathbb{R}^{S^n}$). So take $M_1, M_2 \in \lambda_{\mathcal{B}}(S^n)$ and let $M_3 = \{B(\underline{x},\varepsilon) \mid \underline{x} \in S^n,\ \varepsilon \geq \frac{1}{2}U(M_1)(\underline{x}) + \frac{1}{2}U(M_2)(\underline{x})\}$. Then M_3 is a linked system, because if $B(\underline{x},\varepsilon)$ and $B(\underline{y},\delta) \in M_3$ $(\underline{x},\underline{y} \in S^n,\ \varepsilon \geq \frac{1}{2}U(M_1)(\underline{x}) + \frac{1}{2}U(M_2)(\underline{x}),\ \delta \geq \frac{1}{2}U(M_1)(\underline{y}) + \frac{1}{2}U(M_2)(\underline{y}))$, then:

$$d(\underline{x},\underline{y}) \le U(M_1)(\underline{x}) + U(M_1)(\underline{y}), \text{ and}$$
$$d(\underline{x},\underline{y}) \le U(M_2)(\underline{x}) + U(M_2)(\underline{y}); \text{ hence}$$
$$d(\underline{x},\underline{y}) \le \delta + \varepsilon, \text{ i.e. } B(\underline{x},\varepsilon) \cap B(\underline{y},\delta) \ne \emptyset.$$

Let $\bar{M}_3$ be a maximal linked system containing M_3 (in fact M_3 is itself a maximal linked system). Then, clearly,

$$U(\bar{M}_3)(\underline{x}) \le \tfrac{1}{2}U(M_1)(\underline{x}) + \tfrac{1}{2}U(M_2)(\underline{x}), \text{ and}$$
$$U(\bar{M}_3)(\underline{x}) \le \tfrac{1}{2}U(M_1)(\underline{x}) + \tfrac{1}{2}U(M_2)(\underline{x}), \text{ for each } \underline{x} \in S^n.$$

But, since for each maximal linked system M: $U(M)(\underline{x}) + U(M)(\bar{\underline{x}}) = \pi$ we have

$$U(M_3)(\underline{x}) = \tfrac{1}{2}U(M_1)(\underline{x}) + \tfrac{1}{2}U(M_2)(\underline{x}), \quad \text{for each } \underline{x} \in S^n.$$

Thus

$$U(\bar{M}_3) = \tfrac{1}{2}U(M_1) + \tfrac{1}{2}U(M_2). \quad \square$$

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